NON-LOCALITY AND ELLIPTICITY IN A GAUGE-INVARIANT QUANTIZATION

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Abstract. The quantum theory of a free particle in two dimensions with non-local boundary conditions on a circle is known to lead to surface and bulk states. Such a scheme is here generalized to the quantized Maxwell field, subject to mixed boundary conditions. If the Robin sector is modified by the addition of a pseudo-differential boundary operator, gauge-invariant boundary conditions are obtained at the price of dealing with gauge-field and ghost operators which become pseudo-differential. A good elliptic theory is then obtained if the kernel occurring in the boundary operator obeys certain summability conditions, and it leads to a peculiar form of the asymptotic expansion of the symbol. The cases of ghost operator of negative and positive order are studied within this framework.

1. Introduction

In the late eighties, non-local boundary conditions for operators of Laplace type found some interesting applications to physical problems, i.e. the behaviour of a free Bose gas and the phenomenon of Bose–Einstein condensation. More precisely, the work in Ref. 1 studied the stationary Schrödinger equation for a free particle in polar coordinates on a circle of radius R:

$$-\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]u(r,\varphi)=Eu(r,\varphi) \;\; r<R, \tag{1.1}$$

subject to the boundary condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=R}+R\int_{-\pi}^{\pi}q_R(R(\varphi-\theta))u(R,\theta)d\theta=0. \tag{1.2}$$

In Eq. (1.2), q_R is defined as

$$q_R(x)\equiv\frac{1}{2\pi R}\sum_{l=-\infty}^{\infty}e^{ilx/R}\int_{-\infty}^{\infty}e^{-ily/R}q(y)dy, \tag{1.3}$$

where q is integrable and square-integrable on $\mathbf{R}$: $q\in L_1(\mathbf{R})\cap L_2(\mathbf{R})$. Both positive ($E>0$) and negative ($E<0$) eigenvalues are admissible. In the former case, two sets of eigenfunctions are obtained: surface states, which decrease exponentially with increasing distance from the boundary, and bulk states, corresponding to more extended eigenfunctions. For negative eigenvalues, only surface states are found to occur.

Interestingly, (even) for a free Schrödinger operator, a non-local boundary condition such as the one in (1.2) may lead to two sets of solutions which, in other physical problems, are normally obtained under quite different conditions. We have therefore tried to understand whether the above scheme admits an extension to some fundamental field theory. The first non-trivial example is given, in our opinion, by an Abelian gauge theory, i.e. the free Maxwell field $F_{ab} \equiv \nabla_a A_b - \nabla_b A_a$ in vacuum, where A_b is the electromagnetic potential and ∇ is the Levi–Civita connection of the background geometry. At the classical level, this may be studied by imposing a supplementary condition of Lorenz type:

$$\nabla^b A_b = 0, \tag{1.4}$$

so as to obtain the homogeneous wave equation for the electromagnetic potential. In the quantum theory via path integrals, one performs Gaussian averages over gauge functionals,[2] hereafter denoted by $\Phi(A)$, to avoid "summing" over gauge-equivalent field configurations. This is achieved by adding the term $\frac{[\Phi(A)]^2}{2\alpha}$ to the original Maxwell Lagrangian, where α is a gauge parameter. In particular, in the one-loop semiclassical theory, the resulting gauge-field operator $P_a^{\ b}$ acting on A_b is found to have a non-degenerate leading symbol. This operator is hence elliptic, with a well defined Green function, thanks to the introduction of a term that "breaks" the gauge invariance properties of $\frac{1}{4} F_{ab} F^{ab}$. Moreover, to the gauge functions of the classical theory, for which

$$^f A_b \equiv A_b + \nabla_b f \tag{1.5}$$

leads to the same field equations as A_b, there correspond two fermionic ghost fields in the quantum theory (usually referred to as ghost and anti-ghost, although they are actually independent[2]), with the associated ghost operator. Its "classical counterpart" is clearly obtained if one remarks that the supplementary condition (1.4) is preserved under the action of (1.5) if and only if the gauge function obeys the equation $\Box f = 0$. The form of the equation obeyed by f will depend, of course, on which supplementary condition is chosen.

If one accepts the view that the potential A is more fundamental than the Maxwell field $F = dA$ (this is suggested by the Aharonov–Bohm effect and by the emphasis on differential operators in the path integral), the boundary conditions should also involve the potential and the ghost in the first place (rather than using components of $\vec{E}$ and $\vec{B}$). This formulation, although not compelling, is certainly appropriate if one studies the gauge-field operator $P_a^{\ b}$, since this acts on A_b, and therefore cannot be properly studied without specifying the boundary conditions on A_b. A set of gauge-invariant boundary conditions is obtained upon requiring that the tangential components of A should vanish at the boundary:

$$[A_k]_{\partial M} = 0, \tag{1.6}$$

jointly with the gauge-averaging functional, i.e.

$$[\Phi(A)]_{\partial M} = 0. \tag{1.7}$$

One can in fact prove that both Eq. (1.6) and Eq. (1.7) are preserved under gauge transformations on the potential if the ghost field obeys homogeneous Dirichlet conditions (see Ref. 3 and our Appendix):

$$[\varepsilon]_{\partial M} = 0. \tag{1.8}$$

In particular, if (free) Maxwell theory is quantized in the Lorenz gauge (cf. Eq. (1.4)), equations (1.6) and (1.7) are found to imply a Robin boundary condition on the normal component of the potential, i.e.

$$\left[\frac{\partial A_0}{\partial n} + A_0 \mathrm{Tr} K\right]_{\partial M} = 0, \tag{1.9}$$

where K is the extrinsic-curvature tensor of the boundary.

Section 2 studies a non-local modification of Eq. (1.9) inspired by Eq. (1.2), jointly with the request of gauge invariance of the whole set of boundary conditions. The resulting gauge-field and ghost operators are studied in Sec. 3, and the conditions for an elliptic theory are analyzed in Sec. 4. Concluding remarks are presented in Sec. 5, and relevant details are given in the Appendix.

2. Boundary Conditions: Non-Locality and Gauge Invariance

Since we are interested in a generalization of the model described by Eqs. (1.1)–(1.3) to Maxwell theory expressed in terms of the potential, we are led to modify the Robin sector of the boundary conditions (1.6) and (1.9), by requiring that

$$\left[\frac{\partial A_0}{\partial n} + A_0 \mathrm{Tr} K\right]_{\partial M} + R \int_{-\pi}^{\pi} q_R(R(\varphi - \theta)) A_0(R, \theta) d\theta = 0. \tag{2.1}$$

However, to avoid having a non-local boundary operator which spoils gauge invariance of the boundary conditions, we should be able to regard Eq. (2.1) as a particular case of Eq. (1.7) when the boundary condition (1.6) is imposed. Thus, upon choosing the normal to the boundary in the form $N^b = (1, 0)$, we are led to consider a gauge-averaging functional (hereafter all indices take the values 0 and 1)

$$\Phi(A) \equiv \nabla^b A_b + N^b Q_b, \tag{2.2}$$

where Q_a is defined by

$$Q_a \equiv N_a r \int_{-\pi}^{\pi} q_r(r(\varphi - \theta)) N^b A_b(r, \theta) d\theta, \tag{2.3}$$

which ensures that (see (2.1))

$$[N^b Q_b]_{\partial M} = [Q_0]_{\partial M} = R \int_{-\pi}^{\pi} q_R(R(\varphi - \theta)) A_0(R, \theta) d\theta. \tag{2.4}$$

The full set of boundary conditions is now given by (1.6), (2.1) and the Dirichlet condition (1.8) on the ghost.

3. Ghost and Gauge-Field Operators

The ghost operator is obtained by taking the difference between the gauge-averaging functional $\Phi(A)$ and the same functional when acting on the gauge-transformed potential ${}^{\varepsilon}A_b \equiv A_b + \nabla_b \varepsilon$ (see (1.5)). In our problem, by virtue of the choice (2.2), one finds

$$\Phi(A) - \Phi({}^{\varepsilon}A) = \mathcal{P}\varepsilon, \tag{3.1}$$

where the action of $\mathcal{P}$, the ghost operator, reads

$$\mathcal{P}\varepsilon = -\Box \varepsilon - r \int_{-\pi}^{\pi} q_r(r(\varphi - \theta)) N^c \nabla_c \varepsilon(r, \theta) d\theta, \tag{3.2}$$

having defined $\Box \equiv g^{ac} \nabla_a \nabla_c = \nabla^b \nabla_b$. Such a ghost operator is an integro-differential operator by virtue of the occurrence of Q_b in (2.2).

The corresponding gauge-field operator is also integro-differential, because it is obtained from the "Lagrangian"

$$L \equiv \frac{1}{4} F_{ab} F^{ab} + \frac{[\Phi(A)]^2}{2\alpha}, \tag{3.3}$$

after writing it in the form

$$L = \frac{1}{2} A^a \; P_a^{\ b} \; A_b + \text{total derivative}. \tag{3.4}$$

For this purpose we use the identity

$$\frac{1}{4} F_{ab} F^{ab} = \frac{1}{2} (\nabla_a A_b)(\nabla^a A^b) - \frac{1}{2} (\nabla_a A_b)(\nabla^b A^a), \tag{3.5}$$

the Leibniz rule and the commutator of covariant derivatives to prove that the first term on the right-hand side of (3.3) contributes

$$S_a^{\ b} \equiv -\delta_a^{\ b} \Box + \nabla_a \nabla^b + R_a^{\ b} \tag{3.6}$$

to $P_a^{\ b}$. As is well known, this operator has a degenerate leading symbol[4] and hence is not invertible. To deal with the second term on the right-hand side of (3.3) we use (2.2) and the identities

$$\frac{1}{2\alpha} [\Phi(A)]^2 = \frac{1}{2\alpha} (\nabla^b A_b)(\nabla^c A_c) + \frac{1}{\alpha} (\nabla^b A_b)(N^c Q_c) + \frac{1}{2\alpha} (N^b Q_b)(N^c Q_c), \tag{3.7}$$

$$(\nabla^b A_b)(\nabla^c A_c) = \nabla^b (A_b \nabla^c A_c) - A^a \nabla_a \nabla^b A_b, \tag{3.8}$$

$$(\nabla^b A_b)(N^c Q_c) = \nabla^b (A_b N^c Q_c) - A^a \nabla_a r \int_{-\pi}^{\pi} q_r(r(\varphi - \theta)) N^b A_b(r, \theta) d\theta. \tag{3.9}$$

By virtue of (3.3)–(3.9), the gauge-field operator takes the form

$$P_a^{\ b} = -\delta_a^{\ b} \Box + \left(1 - \frac{1}{\alpha}\right) \nabla_a \nabla^b + R_a^{\ b} + \frac{1}{\alpha} T_a^{\ b} + \frac{1}{\alpha} U_a^{\ b}, \tag{3.10}$$

where ${T_a}^b$ and ${U_a}^b$ are integral operators defined by

$$\left({T_a}^b A_b\right)(r,\varphi) \equiv -2\nabla_a r \int_{-\pi}^{\pi} q_r(r(\varphi-\theta))N^b A_b(r,\theta)d\theta, \tag{3.11}$$

$$\left(A^a {U_a}^b A_b\right)(r,\varphi) \equiv \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} A^a(r,\theta)r^2 q_r(r(\varphi-\theta))q_r(r(\varphi-\theta')) \\ N_a N^b A_b(r,\theta')d\theta d\theta'. \tag{3.12}$$

4. Ellipticity

The compatibility of gauge invariance of the boundary conditions with their non-local character has been shown to lead to integro-differential gauge-field and ghost operators. Before we can regard all this as a viable scheme, some consistency checks are in order. In particular, we are here interested in preserving the ellipticity of the theory, which is known to hold when local boundary conditions of mixed nature are imposed.[4]

For this purpose, we here focus on the ghost operator given in (3.2), because the novel features arising from the pseudo-differential framework are already clear at that stage. This makes it necessary to use the definition of ellipticity for pseudo-differential operators, which is first given on open subsets of $\mathbf{R}^m$ and then extended to deal with changes of coordinates, as is shown in Secs. 1.3.1 and 1.3.2 of Ref. 4. The key steps are as follows.[4]

(i) A linear partial differential operator P of order d can be written in the form

$$P \equiv \sum_{|\alpha|\leq d} a_\alpha(x) D_x^\alpha, \tag{4.1}$$

where (i denotes, as usual, the imaginary unit)

$$|\alpha| \equiv \sum_{k=1}^{m} \alpha_k, \tag{4.2}$$

$$D_x^\alpha \equiv (-i)^{|\alpha|}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}, \tag{4.3}$$

and a_α is a C^∞ function on $\mathbf{R}^m$ for all α. The associated *symbol* is, by definition,

$$p(x,\xi) \equiv \sum_{|\alpha|\leq d} a_\alpha(x)\xi^\alpha, \tag{4.4}$$

i.e. it is obtained by replacing the differential operator D_x^α by the monomial ξ^α. The pair (x,ξ) may be viewed as defining a point of the cotangent bundle of $\mathbf{R}^m$, and the action of P on the elements of the Schwarz space $\mathcal{S}$ of smooth complex-valued functions on $\mathbf{R}^m$ of rapid decrease is given by

$$Pf(x) \equiv \int e^{i(x-y)\cdot\xi} p(x,\xi) f(y) dy d\xi, \tag{4.5}$$

where the $dy = dy_1...dy_m$ and $d\xi = d\xi_1...d\xi_m$ orders of integration cannot be interchanged, since the integral is not absolutely convergent.

(ii) Pseudo-differential operators are instead a more general class of operators whose symbol need not be a polynomial but has suitable regularity properties. More precisely, let S^d be the set of all symbols $p(x,\xi)$ such that[4]

(1) p is smooth in (x,ξ), with compact x support.

(2) For all (α,β), there exist constants $C_{\alpha,\beta}$ for which

$$\left| D_x^\alpha D_\xi^\beta p(x,\xi) \right| \leq C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}, \tag{4.6}$$

for some *real* (not necessarily positive) value of d, where $|\beta| \equiv \sum_{k=1}^m \beta_k$ (see (4.2)). The associated *pseudo-differential operator*, defined on the Schwarz space and taking values in the set of smooth functions on $\mathbf{R}^m$ with compact support:

$$P : \mathcal{S} \rightarrow C_c^\infty(\mathbf{R}^m),$$

is defined in a way formally analogous to Eq. (4.5).

(iii) Let now U be an open subset with compact closure in $\mathbf{R}^m$, and consider an open subset U_1 whose closure $\overline{U}_1$ is properly included into U: $\overline{U}_1 \subset U$. If p is a symbol of order

d on U, it is said to be *elliptic* on U_1 if there exists an open set U_2 which contains $\overline{U}_1$ and positive constants C_i so that

$$|p(x,\xi)|^{-1} \leq C_1(1+|\xi|)^{-d}, \tag{4.7}$$

for $|\xi| \geq C_0$ and $x \in U_2$,[4] where

$$|\xi| \equiv \sqrt{g^{ab}(x)\xi_a \xi_b} = \sqrt{\sum_{k=1}^{m} \xi_k^2}. \tag{4.8}$$

The corresponding operator P is then elliptic.

In our problem, we revert to the use of Cartesian coordinates, so that the above definitions can be immediately applied. Hereafter, (x,y) and (x',y') are coordinates of the points X and X' of $\mathbf{R}^2$, respectively. Q_a is the operator defined by a convolution (cf. Eq. (2.3))

$$Q_a f(x,y) \equiv N_a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(x-x', y-y') f(x',y') dx' dy', \tag{4.9}$$

where the unit normal to the circle has components

$$N_1 = N_x = \frac{x}{\sqrt{x^2+y^2}}, \tag{4.10}$$

$$N_2 = N_y = \frac{y}{\sqrt{x^2+y^2}}. \tag{4.11}$$

The gauge-averaging functional is taken to be (cf. Eq. (2.2))

$$\begin{aligned} \Phi(A) &\equiv \nabla^b A_b + N^b Q_b \\ &= \nabla^b A_b + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(x-x', y-y') N^c A_c(x',y') dx' dy'. \end{aligned} \tag{4.12}$$

Hence the action of the ghost operator reads (cf. Eq. (3.2))

$$\mathcal{P}\varepsilon = -\Box \varepsilon - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(x-x', y-y') N^c \nabla_c \varepsilon(x',y') dx' dy'. \tag{4.13}$$

Recall now that the boundary conditions (1.6) and (1.7) are gauge-invariant if and only if the ghost field obeys homogeneous Dirichlet conditions (1.8) on the boundary. We can therefore use the identity (K is again the extrinsic-curvature tensor of the boundary)

$$Q N^c \nabla_c \varepsilon = \nabla_c (N^c Q \varepsilon) - (\mathrm{Tr} K) Q \varepsilon - N^c (\nabla_c Q) \varepsilon, \tag{4.14}$$

the divergence theorem (here $B_R \equiv \left\{x, y : x^2 + y^2 \leq R^2\right\}$):

$$\int_{B_R} \nabla_c (N^c Q \varepsilon) = \int_{\partial B_R} N^c Q \varepsilon \, d\sigma_c, \tag{4.15}$$

and integration by parts, to cast Eq. (4.13) in the form

$$\mathcal{P} \varepsilon = -\Box \varepsilon + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big[((\mathrm{Tr} K) + N^c \nabla_c) Q \Big] \varepsilon(x', y') dx' dy'. \tag{4.16}$$

Equation (4.16) shows clearly that, on setting

$$W(x, y) \equiv \mathrm{Tr} K + N^c \nabla_c, \tag{4.17}$$

the integral operator on the right-hand side has kernel given by

$$\begin{aligned} \chi(x, y; x', y') &\equiv W(x, y) Q(x - x', y - y') \\ &= (\mathrm{Tr} K) Q(\widetilde{x}, \widetilde{y}) + N_x \frac{\partial Q}{\partial \widetilde{x}} + N_y \frac{\partial Q}{\partial \widetilde{y}}, \end{aligned} \tag{4.18}$$

where $\widetilde{x} \equiv x - x', \widetilde{y} \equiv y - y'$. Thus, the corresponding symbol can be evaluated from Eq. (2.1.36) in Ref. 5:

$$p(x, \xi) = \int e^{-i z \cdot \xi} \chi(x; x - z) dz. \tag{4.19}$$

In our case, Eqs. (4.18) and (4.19) imply that the symbol of the ghost operator $\mathcal{P}$ in Eq. (4.16) is given by

$$\begin{aligned} p(x, y; \xi_1, \xi_2) &= |\xi|^2 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(z_1 \xi_1 + z_2 \xi_2)} \chi(x, y; x - z_1, y - z_2) dz_1 dz_2 \\ &= |\xi|^2 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(z_1 \xi_1 + z_2 \xi_2)} F(x, y, z_1, z_2) dz_1 dz_2, \end{aligned} \tag{4.20}$$

where

$$F(x,y,z_1,z_2) \equiv \left((\mathrm{Tr}K)Q(z_1,z_2) + N_x \left. \frac{\partial Q}{\partial \widetilde{x}} \right|_{z_1,z_2} + N_y \left. \frac{\partial Q}{\partial \widetilde{y}} \right|_{z_1,z_2} \right), \tag{4.21}$$

because $\widetilde{x} = z_1$ when $x' = x - z_1$, and $\widetilde{y} = z_2$ when $y' = y - z_1$. To achieve ellipticity in the interior of B_R we now impose the majorization (4.7), re-expressed in the form

$$|p(x,y;\xi_1,\xi_2)| \geq \widetilde{C}_1 (1+|\xi|)^d, \tag{4.22}$$

for all $|\xi| \geq C_0$, where $\widetilde{C}_1 \equiv C_1^{-1}$. On the other hand, by virtue of (4.20), it is always true that

$$|p(x,y;\xi_1,\xi_2)| \geq \left(|\xi|^2 - \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(z_1\xi_1 + z_2\xi_2)} F(x,y,z_1,z_2) dz_1 dz_2 \right| \right). \tag{4.23}$$

To ensure ellipticity in $B_R - \partial B_R$ it is therefore sufficient to impose that

$$\widetilde{C}_1 (1+|\xi|)^d \leq \left(|\xi|^2 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y,z_1,z_2)| dz_1 dz_2 \right), \tag{4.24}$$

where we have used the majorization

$$\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(z_1\xi_1 + z_2\xi_2)} F(x,y,z_1,z_2) dz_1 dz_2 \right| \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y,z_1,z_2)| dz_1 dz_2, \tag{4.25}$$

to go from (4.23) to (4.24).

For example, if $d < 0$, the majorization (4.24) can lead, for $|\xi| \geq C_0$, to

$$C_0^2 - \widetilde{C}_1 (1+C_0)^d \geq \sup_{x,y \in B_R} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y,z_1,z_2)| dz_1 dz_2, \tag{4.26}$$

which is satisfied if

$$Q(z_1,z_2) \in L^1(\mathbf{R}^2), \tag{4.27}$$

and also (see (4.21))

$$\left.\frac{\partial Q}{\partial \widetilde{x}}\right|_{z_1,z_2} \text{ and } \left.\frac{\partial Q}{\partial \widetilde{y}}\right|_{z_1,z_2} \in L^1(\mathbf{R}^2). \tag{4.28}$$

In particular, the equality of left- and right-hand side can be considered in (4.26), so that the resulting order d of the ghost operator $\mathcal{P}$ can be evaluated in the form

$$d = \frac{\log\Bigl(C_1(C_0^2 - I)\Bigr)}{\log(1 + C_0)}, \tag{4.29}$$

where

$$I \equiv \sup_{x,y\in B_R} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x, y, z_1, z_2)| dz_1 dz_2. \tag{4.30}$$

Thus, this particular order of the ghost operator is negative, in agreement with the assumption leading to (4.26), if $C_1(C_0^2 - I) \in]0, 1[$.

Moreover, strong ellipticity should also be studied.[4,5] For this purpose, following Ref. 5, we assume that the symbol of the ghost operator given in Eq. (4.20) is *polyhomogeneous*, in that it admits an asymptotic expansion of the form

$$p(x, y; \xi_1, \xi_2) \sim \sum_{l=0}^{\infty} p_{d-l}(x, y; \xi_1, \xi_2), \tag{4.31}$$

where each term p_{d-l} has the homogeneity property

$$p_{d-l}(x, y; t\xi_1, t\xi_2) = t^{d-l} p_{d-l}(x, y; \xi_1, \xi_2), \tag{4.32}$$

for $t \geq 1$ and $|\xi| \geq 1$. The *principal symbol* p^0 of the ghost operator is then, by definition,

$$p^0(x, y; \xi_1, \xi_2) \equiv p_d(x, y; \xi_1, \xi_2). \tag{4.33}$$

Strong ellipticity (see comments in Sec. 5) is formulated in terms of the principal symbol, because it requires that[5]

$$\text{Re } p^0(x, y; \xi_1, \xi_2) = \frac{1}{2}\Bigl[p^0(x, y; \xi_1, \xi_2) + p^0(x, y; \xi_1, \xi_2)^*\Bigr] \geq c(x)|\xi|^d I, \tag{4.34}$$

where $x \in B_R, c(x) > 0$ and $|\xi| \geq 1$. In other words, given a positive function c, the product $c(x)|\xi|^d I$ should be always majorized by the real part of the principal symbol of the ghost operator. Indeed, the symbol (4.20) is such that

$$p(x, y; t\xi_1, t\xi_2) = t^2\left(\xi_1^2 + \xi_2^2\right)$$
$$+ t^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(z_1\xi_1+z_2\xi_2)} F\left(x, y, \frac{z_1}{t}, \frac{z_2}{t}\right) dz_1 dz_2. \tag{4.35}$$

By virtue of (4.31), (4.32) and (4.35) we find that the kernel Q should have an asymptotic expansion such that

$$t^2|\xi|^2 + t^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(z_1\xi_1+z_2\xi_2)} F\left(x, y, \frac{z_1}{t}, \frac{z_2}{t}\right) dz_1 dz_2$$
$$\sim \sum_{l=0}^{\infty} t^{d-l} p_{d-l}(x, y; \xi_1, \xi_2). \tag{4.36}$$

Moreover, the term on the right-hand side of (4.36) with $l = 0$ should be the one occurring in the condition (4.34) for strong ellipticity. Our understanding of the necessary class of kernels has therefore made progress.

Last, if the order of the ghost operator $\mathcal{P}$ is positive and even, one can use Theorem 1.7.2 in Ref. 5 to prove ellipticity with Dirichlet boundary conditions as in Eq. (1.8). Since we have previously discussed the case of negative order, it is necessary to describe how such a positive order can be obtained. To begin note that, if f is a C^∞ function on $\mathbf{R}^m$ with compact support, one can define a symbol of order d for any $d \in \mathbf{R}$ by using the formula[4]

$$p(x, \xi) \equiv f(x)(1 + |\xi|^2)^{\frac{d}{2}}. \tag{4.37}$$

The associated kernel is then[5]

$$\chi(x, y) = (2\pi)^{-m} \int e^{i(x-y)\cdot\xi} p(x, \xi) d\xi. \tag{4.38}$$

Consider now, for simplicity, the case $m = 1$ and $d = 4$. Such formulae make it then necessary to evaluate the kernel χ by studying the integral (hereafter $X \equiv x - y$)

$$J(X) \equiv \int_{-\infty}^{\infty} e^{iX\xi}(1 + 2\xi^2 + \xi^4)d\xi, \tag{4.39}$$

which is meaningful within the framework of Fourier transforms of distributions.[6] To get an explicit representation, we write $J(X)$ in the form

$$J(X) \equiv \int_{-\infty}^{\infty} \lim_{a \to 0} e^{-a\xi^2} e^{iX\xi}(1 + 2\xi^2 + \xi^4)d\xi, \tag{4.40}$$

and hence consider the parameter-dependent integrals

$$J_{1,a}(X) \equiv \int_{-\infty}^{\infty} e^{-a\xi^2} e^{iX\xi} d\xi = \sqrt{\frac{\pi}{a}} e^{-X^2/4a}, \tag{4.41}$$

$$J_{2,a}(X) \equiv 2\int_{-\infty}^{\infty} \xi^2 e^{-a\xi^2} e^{iX\xi} d\xi = \frac{1}{a}\sqrt{\frac{\pi}{a}} e^{-X^2/4a}\left(1 - \frac{X^2}{2a}\right), \tag{4.42}$$

$$\begin{aligned} J_{3,a}(X) &\equiv \int_{-\infty}^{\infty} \xi^4 e^{-a\xi^2} e^{iX\xi} d\xi \\ &= \frac{3}{4a^2}\sqrt{\frac{\pi}{a}} e^{-X^2/4a}\left[1 - \frac{X^2}{a} + \frac{1}{12}\frac{X^4}{a^2}\right]. \end{aligned} \tag{4.43}$$

The link with the theory of distributions is now clear if one bears in mind that the following regular distribution:

$$\frac{\nu}{\sqrt{\pi}} e^{-\nu^2\tau^2}$$

converges in the space of all continuous linear functionals to the delta functional as $\nu \to \infty$. If one sets $\nu = \frac{1}{\sqrt{a}}, X = 2\tau$, the integral $J_{1,a}$ converges therefore to π times the delta functional as $a \to 0$. One can treat similarly $J_{2,a}$ and $J_{3,a}$, and hence prove explicitly the distributional nature of the kernel χ obtained from (4.37)–(4.39).

To sum up, positive orders of the ghost operator, which are necessary to prove ellipticity with Dirichlet boundary conditions, lead to a kernel $\chi(x, y; x', y')$ (see (4.18)) of distributional nature. An explicit representation can be obtained by an m-dimensional generalization of the integrals (4.41)–(4.43). Moreover, the kernel $Q(x-x', y-y')$ which contributes to $\chi(x,y;x',y')$ should be such that the asymptotic expansion (4.36) holds. This makes it possible to pick out the principal symbol which occurs in the strong ellipticity condition (4.36).

5. Concluding Remarks

In the first part of our paper we have shown how to choose non-local boundary conditions for the quantized Maxwell field in a way compatible with the request of complete gauge invariance of the resulting boundary operator. This scheme has been found to lead to gauge-field and ghost operators of integro-differential nature. In the second part of our paper we have studied more carefully the ghost operator within the framework of pseudo-differential operators on $\mathbf{R}^2$. Interestingly, such an operator remains elliptic in the interior of the region considered therein provided that the kernel occurring in the boundary operator fulfills the summability conditions (4.27) and (4.28). Moreover, strong ellipticity for the ghost holds if (4.36) and (4.34) are satisfied. The above results are, to our knowledge, completely new in the physical literature, although some non-local aspects in the quantization of Maxwell theory had been studied, for example, in Refs. 7 and 8. The ultimate meaning of strong ellipticity is that it ensures the existence of the asymptotic expansion of the L^2-trace of the heat semigroup associated to the given operator,[4] so that the resulting conformal anomaly is well defined in one-loop quantum theory. From the mathematical point of view, strong ellipticity is a precise formulation of existence and uniqueness of smooth solutions with given boundary conditions in an elliptic boundary-value problem.[4]

As far as physics is concerned, it now appears crucial to understand which novel features of quantized gauge theories can result from the consideration of pseudo-differential boundary-value problems. Last, but not least, such investigations might have a non-trivial impact on the attempts of quantizing the gravitational field, where the role of a non-local formulation[9,10] is also receiving careful consideration.

Appendix

Since not all readers might be familiar with boundary conditions on the potential and the ghost field for Maxwell theory, we find it appropriate to prove why the ghost has to obey homogeneous Dirichlet conditions in our problem.

If tangential components of the potential are set to zero at the boundary as in Eq. (1.6), the preservation of this part of the boundary conditions under gauge transformations leads to

$$[\partial_k \varepsilon]_{\partial M} = 0. \tag{A.1}$$

But tangential derivatives only act on the part of ε depending on the local coordinates on the $(m-1)$-dimensional boundary, and hence, if Eq. (1.8) is imposed, Eq. (A.1) is automatically satisfied. In other words, the operations of tangential derivative and restriction to the boundary turn out to commute.

Moreover, if $\mathcal{P}$ is symmetric and elliptic, the field ε can be expanded in a complete orthonormal set of C^∞ eigenvectors ε_λ of $\mathcal{P}$, for which

$$\mathcal{P}\varepsilon_\lambda = \lambda \varepsilon_\lambda. \tag{A.2}$$

In other words, one can write

$$\varepsilon = \sum_\lambda C_\lambda \varepsilon_\lambda, \tag{A.3}$$

which implies, by virtue of Eq. (3.1),

$$\Phi(A) - \Phi({}^{\varepsilon}A) = \sum_\lambda \lambda C_\lambda \varepsilon_\lambda. \tag{A.4}$$

Thus, if Eq. (1.8) holds, which is satisfied when

$$[\varepsilon_\lambda]_{\partial M} = 0 \;\; \forall \, \lambda, \tag{A.5}$$

then the vanishing of the gauge-averaging functional at the boundary is a gauge-invariant boundary condition as well on the remaining part of the potential (we are ruling out the occurrence of zero-modes, i.e. non-vanishing eigenvectors ε_λ belonging to the zero eigenvalue $\lambda = 0$). Thus, the boundary conditions (1.6) and (1.7) are both gauge invariant under the same condition on the ghost if Eq. (1.8) is satisfied.

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